DIPARTIMENTO DI MATEMATICA

Prova di verifica della preparazione per l'ammissione

alla Laurea Magistrale in Matematica

Written Exam for the Admission to the Master degree in Mathematics

11 luglio 2023

Part common to all curricula

Linear Algebra - one of the following two exercises.

Exercise 1. *Consider the following linear system which depends on the parameter* $k \in \mathbb{R}$

$$\begin{cases} x - y + kz &= 2 \\ x - y + 4z &= k - 2 \\ 3x + ky - z &= 1 \end{cases}$$

a) *Discuss the existence of solutions for the system as* k *varies in* $\mathbb{R}$.
b) *For the values of* k *for which the system is consistent, find its solutions.*
c) *Is the set of solutions a subspace of* $\mathbb{R}^3$ *for some* k*?*

Exercise 2. *Let*

$$S = \begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix}$$

and let $T : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ *be the map that associates to a matrix* A *the matrix* SA. *[*$M_2(\mathbb{R})$ *is the space of the* 2×2 *real matrices.]*

a) *Verify that* T *is a linear map.*
b) *Write a representative matrix of* T.
c) *Show that the identity matrix is not in the image of* T.

Mathematical Analysis - one of the following two exercises.

Exercise 3. *For* $\alpha \in \mathbb{R}$, *let* $f_\alpha : \mathbb{R} \to \mathbb{R}$ *the function defined by*

$$f_\alpha(x) = \begin{cases} \dfrac{(1-x^3)\sqrt{x}}{e^{2\sqrt{x}} - 1} & \text{se } x > 0 \\ x + \alpha & \text{se } x \le 0, \end{cases}$$

and let

$$F_\alpha(x) = \int_0^x f_\alpha(t)\,\mathrm{d}t$$

its integral function.

(1) *Explain why the function* F_α *is well defined and continuous for each* $x \in \mathbb{R}$.
(2) *Say whether the limit* $\lim_{x \to +\infty} F_\alpha(x)$ *exists, and whether it is finite or infinite.*
(3) *Determine the values of the parameter* α *such that* F_α *is differentiable in* $x = 0$.
(4) *For the value of* α *determined in the previous point, find the possible local maximum/minimum points of* F_α.
Plot a qualitative graph of F_α.

Exercise 4. *Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ two sequences of positive real numbers such that*

$$a_n = o(b_n) \qquad n \to +\infty.$$

[Remember that, if f and g are two functions defined in a neighbourhood of c and $g(x) \neq 0$ for $x \neq c$, $f = o(g)$ for $x \to c$ means $\lim_{x\to c} \frac{f(x)}{g(x)} = 0$, and an analogous definition holds for sequences/]

Establish for each of the following statements whether it follows from the hypotheses. Motivate your answers, providing, if possible, a counterexample.

(1) *If $\lim_{n\to\infty} a_n = 0$, then $\lim_{n\to\infty} b_n = 0$.*

(2) *If the series $\sum_{n=0}^{+\infty} b_n$ is convergent, then the series $\sum_{n=0}^{+\infty} a_n$ is also convergent.*

(3) *If the series $\sum_{n=0}^{+\infty} a_n$ is convergent, then the series $\sum_{n=0}^{+\infty} b_n$ is also convergent.*

(4) *If $a_n b_{n+1} \geq a_{n+1} b_n$ for every $n \in \mathbb{N}$, then the series $\sum_{n=0}^{+\infty} (-1)^n \frac{a_n}{b_n}$ is convergent.*

Curriculum Mathematics and Statistics
for Life and Social Sciences

Solve two between the following 3 problems:

Exercise 1. *We are given a coin and assured that it is fair.*

(1) *We flip the coin 64 times, and get 25 heads. What is the probability of such an event?*
(2) *We continue to toss the coin and, out of 256 trials, 100 heads have been obtained. On the basis of these data, what is a point estimate for the probability p that one gets Head when tossing the coin?*
(3) *Based on the data obtained in the previous point, can we reject, at the 5% significance level, the hypothesis that the coin is fair?*
(4) *How can we compute the p-value of the data? [We expect to see a formula, not a precise number]*

Exercise 2.

(i) *Describe Euler's method for the approximation of the solution of a Cauchy problem of the form*

$$\begin{cases} y'(t) &= f(t,y)\,, \quad t \in [t_0, t_0+T]\,, \\ y(t_0) &= y_0\,. \end{cases}$$

(ii) *What is the order of convergence of Euler's method? Provide a definition of convergence order.*
(iii) *For the Cauchy problem*

$$\begin{cases} y'(t) &= -t^2 y\,, \quad t \in [0,1]\,, \\ y(0) &= 1\,, \end{cases}$$

approximate $y(1)$ using Euler's method with step $h = 0.5$.

Exercise 3. *Consider the system of differential equations*

$$\begin{cases} x'(t) &= 10x(t)\left(1 - \dfrac{x(t)}{4}\right) - x(t)y(t) \\ y'(t) &= \dfrac{1}{2}x(t)y(t) - y(t) \end{cases} .$$

(1) *Find all equilibria of this equation. Verify that among them, there is one (which we denote E^*) with both positive coordinates.*
(2) *Study the stability of the equilibrium E^*.*
(3) *What is the approximate behavior of the solutions of the equation starting from an initial value $(x(0), y(0))$ close to E^*?*

Curriculum Teaching and Scientific Communication

In this exam session, there are two possible choices

- Solve two exercises among those of mathematics, and answer the multiple choice questions of physics;
- Solve three exercises among those of mathematics.

Mathematics exercises

Algebra

Exercise 1. *Let* $\mathbb{R}$ *be the field of real numbers, and* $\mathbb{R}^*$ *the set of real numbers different from zero. In* $G = \mathbb{R}^* \times \mathbb{R}$ *define*

$$(a, b) * (c, d) := (ac, ad + b)$$

Prove that $(G, *)$ *is a nonabelian group.*

Geometry

Exercise 2. *Let* $\mathbf{P} : \mathbb{R} \to \mathbb{R}^3$ *be the circular helix defined by*

$$\mathbf{P}(s) = \frac{\sqrt{2}}{2}(\cos(s), \sin(s), s).$$

and let γ *be the curve defined by the endpoints of the unit tangent vectors of* $\mathbf{P}$*. Compute the curvature of* γ*. show that* γ *is a circular helix, too.*

Functions of several real variables

Exercise 3. *Provide an example of function* $f : \mathbb{R}^2 \to \mathbb{R}$ *such that all directional derivatives of* f *exist at* $(0, 0)$ *but* f *is still discontinuous at* $(0, 0)$*.*

Ordinary differential equations

Exercise 4. *Write the first order ordinary differential system equivalent to the second order ordinary differential equation* $y''(x) + x\ln(1 + y(x)^2) = y'(x)$*.*

Topology

Exercise 5. *Determine the interior, the closure and the boundary of the set*

$$\left\{ \left(1 - \frac{1}{k}\right) \left(\cos\left(\frac{k\pi}{2}\right), \sin\left(\frac{k\pi}{2}\right)\right) \,\middle|\, k = 1, 2, \ldots \right\}.$$

Exercise 6. *By placing two springs with elastic constants k_1 and k_2 in series, one obtains an equivalent spring with an elastic constant given by*

(a) $k_1 + k_2$;

(b) $k_1 k_2$;

(c) $1/(1/k_1 + 1/k_2)$;

(d) k_1/k_2.

Briefly justify your chosen answer.

Exercise 7. *Which of the following transformations of a gas is necessarily reversible?*

(a) *Adiabatic and isochoric;*

(b) *Isothermal and isobaric;*

(c) *Isochoric;*

(d) *Adiabatic.*

Briefly justify your chosen answer.